Graz Economics Papers – GEP

GEP 2019–03

# Limit Orders under Knightian Uncertainty

Michael Greinecker and Christoph Kuzmics

March 2019

Department of Economics
Department of Public Economics
University of Graz

# Limit Orders under Knightian Uncertainty*

Michael Greinecker[†] Christoph Kuzmics[‡]

March 1, 2019

**Abstract**

Investors who maximize subjective expected utility will generally trade in an asset unless the market price exactly equals the expected return, but few people participate in the stock market. [Dow and da Costa Werlang, *Econometrica* 1992] show that an ambiguity averse decision maker might abstain from trading in an asset for a wide interval of prices and use this fact to explain the lack of participation in the stock market. We show that when markets operate via limit orders, all investment behavior will be observationally equivalent to maximizing subjective expected utility; ambiguity aversion has no additional explanatory power.

## Introduction

Consider a decision maker who can buy or (short) sell an asset. If the decision maker acts according to the Gilboa and Schmeidler (1989) multiple prior

*Examples similar to our motivating example have been studied by Dominique Päper and Peter Habiger in their respective master theses written under the supervision of Christoph Kuzmics. We are grateful to both as well as Patrick Beissner, Frank Riedel, Jan Werner, and Michael Zierhut as well as seminar audiences at the University of Bielefeld, Caltech, UC Irvine, University of Minnesota, Vienna University of Business and Economics, and participants of the VII Hurwicz workshop in Warsaw for valuable comments and suggestions.

[†]University of Graz, Department of Economics, email: michael.greinecker@uni-graz.at
[‡]University of Graz, Department of Economics, email: christoph.kuzmics@uni-graz.at

model, different probabilities will, in general, be used to calculate the gains from buying and selling, respectively, and the resulting wedge in expected returns can allow for both buying and selling to be worse than abstaining from any trade. There can be nontrivial intervals of prices at which the decision maker is not willing to trade. This cannot happen with a standard subjective expected utility maximizer. Dow and da Costa Werlang (1992) have argued that ambiguity aversion[1] can, therefore, explain why many people refrain from trading stocks.

We show that this result hinges on how exactly trading is modeled. If decision makers submit (generalized) limit orders, no behavior can arise in standard models of ambiguity averse decision makers that cannot occur under subjective expected utility maximization. We only rely on the requirement, satisfied by all decision theories satisfying a weak monotonicity condition, that a decision maker makes no strictly dominated choices.[2]

The model of Dow and da Costa Werlang (1992) is given as the leading example in Gilboa, Postlewaite, and Schmeidler (2008) and Gilboa and Marinacci (2013) for why the Gilboa and Schmeidler (1989) multiple prior model leads to economically significantly different behavior than any subjective expected utility model. Easley and O'Hara (2009) examine the consequences for a number of issues arising in the regulation of financial markets.

The portfolio inertia implicit in the Dow and da Costa Werlang (1992) model also appears in the dynamic asset pricing model of Epstein and Wang (1994). An important implication for equilibrium theory is that supporting prices are robustly not unique and therefore equilibria locally indeterminate. Rigotti and Shannon (2012) show, however, that equilibria in such general equilibrium models with both risk and uncertainty are generically

[1]Not all models of ambiguity averse decision making will give rise to this form of nonparticipation; see the discussion in Epstein and Schneider (2010).

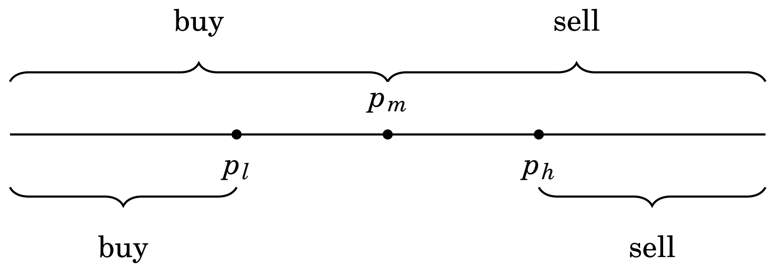
[2]These include, among others, the maxmin expected utility model of Gilboa and Schmeidler (1989), the Choquet expected utility model of Schmeidler (1989), the smooth ambiguity model of Klibanoff et al. (2005), the variational and multiplier preference models of Maccheroni et al. (2006) and Hansen and Sargent (2001), confidence function preferences of Chateauneuf and Faro (2009), uncertainty aversion preferences of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), and the incomplete preference model of Bewley (2002).

determinate. However indeterminacy is robust if preferences are of the Bewley (2002) incomplete type, see Rigotti and Shannon (2005), or there is uncertainty in the price system as in Beissner and Riedel (2019).

## Motivating Example

We illustrate our central point in terms of the leading example of Dow and da Costa Werlang (1992). There is one asset whose future value can be either $-1$ or 1. A risk-neutral decision-maker is faced with a price of $p$ and asked to either buy one unit of the asset, (short) sell one unit of the asset, or not trade at all. If the decision maker maximizes subjective expected utility with respect to some belief $\rho$ on the value being 1, not trading can only be optimal if $\rho 1+(1-\rho)(-1)=p$. Suppose now that the decision maker behaves according to the multiple prior model of Gilboa and Schmeidler (1989) and has two priors $\rho_l < \rho_h$ in their set of priors. There is a whole interval at which it is optimal not to trade; buying is only optimal, if $\rho_l-(1-\rho_l)\geq p$ and selling is only optimal if $\rho_h-(1-\rho_h)\leq p$.

We now consider a situation in which the decision maker has to submit a (generalized) limit order before prices realize. To mimic the economic behavior from before, the decision maker must choose a limit order such that for some $p_l < p_h$ the limit order prescribes buying for prices below $p_l$, selling for prices above $p_h$, and not trading for prices in between. To evaluate the returns from such a limit order, the decision maker needs a model of price formation. Sticking closely to the example of Dow and da Costa Werlang (1992), we assume that prices are not informative regarding the future value of the asset. Prices are distributed according to a full support distribution without mass points on the real line. We show that the mentioned limit order is always strictly dominated by a simple threshold limit order that prescribes selling above a given price $p_m$ and buying below this price. To do so, take $p_m$ to be the unique point between $p_l$ and $p_h$ such that the probability of prices being in the intervals $[p_l,p_m]$ and $[p_m,p_h]$ is the same; in other words $p_m$ is the median price conditional on the price falling in the interval $[p_l,p_h]$.




Any payoff difference comes from how the limit orders behave in the interval $[p_l, p_h]$. The original limit order generates an expected return of zero on this interval under both priors. The new threshold limit order will, conditional on the price being in the interval $[p_l, p_h]$, buy with probability 1/2 at the comparatively low prices between $p_l$ and $p_m$ and with probability 1/2 sell at the comparatively high price between $p_m$ and $p_h$. This leads to an additional positive return that is independent of the priors; the original limit order is strictly dominated. Another way to see this is to consider the possibility that the decision maker can randomize and buy and sell with probability 1/2 each for each price in $[p_l, p_h]$. The expected surplus would still be zero in the interval independent of the prior. But the threshold limit order we constructed also buys and sells with probability 1/2 in the interval but buys when prices are low and sells when prices are high. This generates an additional expected return and requires no randomization. While our argument here only shows that a certain interval limit order is dominated, it follows from Theorem 3 below that only threshold limit orders are undominated in this example.

A peculiar, and somewhat unrealistic, property of the example above is that prices are not informative about the value of the asset. If the decision maker has some models in mind of how prices and values are (stochastically) related, not trading can even be optimal for someone maximizing subjective expected utility. This is the case if and only if the expected value at a price, averaged over all models according to a probabilistic belief, equals that price. At this price the market is efficient in this averaged model. Ambiguity aversion cannot supply additional reasons for nonparticipation; this is the content of Theorem 2.

The key insight needed to prove Theorems 2 and 3 is this. Every limit order that cannot be rationalized as subjective expected utility maximization must be strictly dominated; this is the content of Lemma 1. Moreover, and crucially, it must be strictly dominated by a non-randomized limit order; this follows from Theorem 1. This difference matters. Indeed, Raiffa (1961) already pointed out that in the thought experiments of Ellsberg (1961), ambiguity avoiding choices are strictly dominated by behavior that conditions on randomization devices such as coins. This insight holds outside the thought experiments of Ellsberg: As a consequence of standard dominance characterizations that have roots in classical decision theory Kuzmics (2017) has recently shown in the Anscombe and Aumann (1963) framework that any choices that cannot be rationalized as subjective expected utility maximization must be strictly dominated by a randomized choice.[3] Our Lemma 1 is a version of the same result in our setting.

While according to most standard (normative) decision theories (all that satisfy a weak monotonicity condition; see Footnote 2), a decision maker cannot make a choice that is dominated by a deterministic choice, it is less clear whether such a decision maker can make a choice that is dominated only by a random choice. After having thrown a coin, a decision-maker is in the same situation as before, only one coin flip richer. In order to use randomization, an ambiguity averse decision maker needs a way to commit to randomization.[4] In the more special setting of our model, randomization by limit orders can be approximated by deterministic limit orders. If a limit order is dominated by some randomized limit order, there must be a deterministic limit order close by in an appropriate topology that also dominates the initial limit order. For this reason, not being dominated by a deterministic limit order is the same as not being dominated by a randomized limit

[3] See, among others, Wald (1947), Pearce (1984), and Battigalli et al. (2016) for characterizations of dominance.

[4] In the absence of commitment Saito (2015) and Ke and Zhang (2017) discuss possible ways of evaluating random choices. A discussion of this issue has also lead to the preference model in Seo (2009). See also Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2017), Eichberger, Grant, and Kelsey (2016) and Kuzmics (2017) for a discussion of this commitment issue.

order; this is our Theorem 1. So standard models of ambiguity aversion do not allow for choices of limit orders that are dominated by deterministic limit orders and subjective expected utility maximizers only exclude limit orders dominated by randomized limit orders. In our setting, there is no difference between these forms of domination, so only choices compatible with subjective expected utility maximization are possible.

# Model and Results

The decision maker faces uncertainty over which probabilistic model best describes the relationship between the market price of an asset and its payoff. These probabilistic models are given by a family of density functions, parametrized by a space $Y$. More precisely, we assume there is a compact metrizable parameter set $Y$ and a bounded nonnegative measurable function $h : \mathbb{R} \times \mathbb{R} \times Y$ continuous in $Y$ such that

$$\int \int h(p,x,y)\ \mathrm{d}p\ \mathrm{d}x = 1$$

for all $y \in Y$ and that there is a nonnegative measurable function $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfying $\int \int d\ \mathrm{d}p\ \mathrm{d}x < \infty$ such that

$$|p-x|\ h(p,x,y) \leq d(p,x)$$

for all $p,x \in \mathbb{R}$ and $y \in Y$.[5] The main restriction imposed by the existence of the dominating integrable function $d$ is that the tails of all distributions vanish sufficiently fast in a uniform way. For example, $h$ could be the density of a bivariate normal distribution or bivariate t-distribution with degrees of freedom $2+\epsilon$ (with $\epsilon > 0$ arbitrarily small), and $Y$ a compact set of pairs of means and invertible covariance matrices.[6] A possible choice of the bound-

[5]The double integrals on $\mathbb{R}\times\mathbb{R}$ are taken with respect to two-dimensional Lebesgue measure, but our proofs shows that we could use any product of $\sigma$-finite measures as long as the first factor is nonatomic. All measurability conditions are with respect to the respective Borel $\sigma$-algebras.

[6]These examples include therefore bivariate distributions with existing means and variance but not necessarily existing kurtosis. In other words, these examples include bivariate

ing function $d$ for all these cases would be a scaled up joint density of two independent t-distributions with degree of freedom strictly between 1 and $1+\epsilon$.

In contrast to Dow and da Costa Werlang (1992), we assume the decision maker acts before prices are known and chooses a limit order. A *limit order* is a measurable function from $\mathbb{R}$ into the set $A = \{b, s, n\}$ (buy, sell, do nothing.) More precisely, limit orders are taken to be equivalence classes of measurable functions from $\mathbb{R}$ to $A$ with two such functions being equivalent if they agree outside a set of measure zero. We denote the set of limit orders by $L$ and endow it with the topology of convergence in measure.[7] We embed $L$ in the space of *mixed limit orders* $\Delta(L)$ via point masses. We generally allow our decision maker to choose mixed limit orders, but an essential part of our argument is that this really makes no difference. There is a payoff function $v : \mathbb{R} \times A \times \mathbb{R} \to \mathbb{R}$ given by

$$v(p,a,x) = \begin{cases} x - p \text{ for } a = b, \\ p - x \text{ for } a = s \\ 0 \text{ for } a = n. \end{cases}$$

The expected payoff from $\mu \in \Delta(L)$ if the parameter is $y$ is

$$V(\mu, y) = \int\int\int v\big(p, l(p), x\big) h(x,p,y)\, \mathrm{d}p\, \mathrm{d}x\, \mathrm{d}\mu(l).$$

The restrictions to risk neutrality and the limited trading options are inessential to our argument; see our discussion at the end. If we take the state space to be $Y$ and the set of outcomes to be the financial gains in $\mathbb{R}$, each limit order $l$ defines an Anscombe-Aumann act that maps the state $y$

distributions with very heavy tails.

[7] Since we are working with Lebesgue measure and not a probability measure, we define convergence in measure so that a sequence of limit orders converges in measure to a limit order $l$ if every subsequence has a further subsequence converging almost everywhere to the limit order $l$. This topology depends only on which sets have measure zero. The resulting topology on $L$ is Polish. Indeed, it follows Dudley (2002, Theorems 9.2.1 to 9.2.3) that this defines a completely metrizable space. Let $\mathscr{C}$ be a countable algebra generating the Borel $\sigma$-algebra on $\mathbb{R}$. The countable family of $\mathscr{C}$-measurable limit orders is a countable dense subset of $L$, so $L$ is separable.

to the probability distribution on $\mathbb{R}$ that assigns to each Borel set $B \subseteq \mathbb{R}$ the probability

$$\int \int 1_{\{(p,x)|v(p,l(p),x)\in B\}} h(p,x,y)\ \mathrm{d}p\ \mathrm{d}x.$$

Our setting, therefore, fits within the usual Anscombe-Aumann framework that forms the basis for the development of most models of ambiguity aversion.

We say that $\mu' \in \Delta(L)$ *strictly dominates* $\mu \in \Delta(L)$ if $V(\mu', y) > V(\mu, y)$ for all $y \in Y$. A mixed limit order that is not strictly dominated by another mixed limit order is *undominated*. A mixed limit order that is not strictly dominated by a (nonrandomized) limit order is *deterministically undominated*. The following lemma gives a version of the familiar characterization of undominated choices from statistical decision theory and game theory.

**Lemma 1.** *A mixed limit order $\mu$ is undominated if and only if there exists a probability distribution $\rho \in \Delta(Y)$ such that $\mu$ is a maximizer of $\int V(\cdot, y)\ \mathrm{d}\rho(y)$.*

As discussed before, ambiguity aversion generates a demand for a commitment to randomization. But in the present setting, commitment to approximate randomization is always possible and we can use this to show that a mixed limit order dominated by another mixed limit order must already be dominated by a deterministic limit order.

**Theorem 1.** *A mixed limit order $\mu$ is undominated if and only if it is deterministically undominated.*

Theorem 1 is our central conceptual result. Next, we use it to show that one can rationalize nonparticipation with ambiguity averse preferences only if one can already rationalize nonparticipation for a subjective expected utility maximizer. A subjective expected utility maximizer will in general only abstain from trading if they believe that, on average over the models considered, the expected payoff equals the price. This is slightly more general than saying that each model (each distribution induced by some $y \in Y$) satisfies the efficient market hypothesis at that price.

For each $\rho \in \Delta(Y)$, we let $\rho_P$ be the marginal with respect to the first coordinate of the probability measure on $\mathbb{R} \times \mathbb{R} \times Y$ with density $h$ under $\lambda \otimes$

$\lambda \otimes \rho$ with $\lambda$ being Lebesgue measure. We call

$$M_\rho = \left\{ p \in \mathbb{R} \middle| \int \int |p - x| h(p,x,y) \, \mathrm{d}x \, \mathrm{d}\rho(y) = 0 \right\}$$

the *martingale part* of $\rho$.

**Theorem 2.** *Let $\mu$ be an undominated mixed limit order. Then there exists $\rho \in \Delta(Y)$ such that $\rho_P\left(l^{-1}(n) \setminus M_\rho\right) = 0$ for $\mu$ almost all $l$.*

The belief $\rho$ is exactly a belief a subjective expected utility maximizer might hold to justify nonparticipation.

Implicit in the setting of Dow and da Costa Werlang (1992) is that prices are not informative in terms of payoffs. To represent this, we now assume that prices are not a signal of payoffs or models. In that case, we recover the insight from our example that all trading behavior must be determined by a threshold. A *threshold limit order* is a limit order $l$ such that for some *threshold* $t \in \mathbb{R}$, $l(p) = b$ if $p \leq t$ and $l(p) = s$ if $p > t$.

**Theorem 3.** *Suppose there exists a strictly positive measurable function $f : \mathbb{R} \to \mathbb{R}$ and a measurable function $g : \mathbb{R} \times Y \to \mathbb{R}$ such that $h(p,x,y) = f(p)g(x,y)$ for all $p,x,y$. Then every undominated mixed limit order is a threshold limit order.*

The strict positivity of $f$ in Theorem 3 guarantees that we introduce no new null sets in the price distribution.

# Discussion

The most obvious restrictions are that our decision maker is risk neutral and can trade at most one unit of the asset. The argument is essentially unchanged if the decision maker is weakly risk-averse and at a wealth level at which the felicity function is differentiable (almost every wealth level will do by Rademacher's theorem). The decision maker is then locally risk neutral. The set of trading opportunities need not be finite but should be a compact interval. If there are limits to borrowing and short-selling, this will automatically be satisfied. In particular, nothing in the proofs of Lemma 1 and Theorem 1 depends on $A$ being finite; they only depend on $A$ being compact metrizable and $v$ being continuous. The remaining proofs require only straightforward modifications.

The formalism we use deserves some discussion. For the proof of Lemma 1, we need expected payoffs to be jointly continuous in mixed limit orders and beliefs over $Y$. Without some restriction of the dependence between the distributions of prices and payoffs, this will generally not be possible. The approach we have taken is inspired by the existence results for Bayesian games of Milgrom and Weber (1985) and Balder (1988). Stinchcombe (2011) discusses discontinuities that can arise when one disposes of the "diffuseness condition" that we employ implicitly by requiring the joint distribution of prices and payoffs to be absolutely continuous with respect to a product measure.

Our formalism uses randomized limit orders. The commonly employed simplification of the approach of Anscombe and Aumann (1963) due to Fishburn (1970) does not allow for randomization over "acts." However, Theorem 1 shows that we do not need any randomization over limit orders. Mixed limit orders can be taken to be ancillary mathematical objects; they are not essential to the main conclusion.

In general, there are problems with randomizing over measurable functions as Aumann (1961) showed. The problem there is that pointwise evaluation of measurable functions is in general not measurable, no matter the $\sigma$-algebra one puts on the space of measurable functions. But limit orders are really equivalence classes of measurable functions that are not evaluated pointwise but by integration. The problems Aumann (1961) raised do, therefore, not affect our arguments.

# Proofs

In what follows, we replace the product Lebesgue measure $\lambda \otimes \lambda$ on $\mathbb{R} \times \mathbb{R}$ by the product of two probability measures $\pi \otimes \xi$ that are both mutually absolutely continuous with respect to $\lambda$. As we will, see all we really need is that $\pi$ is atomless. The change of the underlying measures will not affect the validity of our assumptions. Let $r_\pi$ be a nonnegative Radon-Nikodym derivative of $\lambda$ with respect to $\pi$ and $r_\xi$ be a nonnegative Radon-Nikodym derivative of $\lambda$ with respect to $\xi$. Let $d' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $d'(p,x) = d(p,x) r_\pi(p) r_\xi(x)$ and let $h' : \mathbb{R} \times \mathbb{R} \times Y \to \mathbb{R}$ be given by $h'(p,x,y) = h(p,x,y) r_\pi(p) r_\xi(x)$. Then

$$\int\int d(p,x)\,\mathrm{d}p\,\mathrm{d}x = \int d(p,x)\,\mathrm{d}\lambda\otimes\lambda(p,x) = \int r_\pi(p)\int d(p,x) r_\xi(x)\,\mathrm{d}\xi(x)\,\mathrm{d}\pi(p)$$

$$= \int r_\pi(p) r_\xi(p) d(p,x)\,\mathrm{d}\pi\otimes\xi = \int d'\,\mathrm{d}\pi\otimes\xi(p,x),$$

so $d'$ is $\pi\otimes\xi$-integrable. Clearly, $h'(p,x,y) \leq d'(p,x)$ for all $y \in Y$. Also, we have by a similar argument that $\int h'(p,x,y)\,\pi\otimes\xi(p,x) = \int h(p,x,y)\,\mathrm{d}\lambda\otimes\lambda(p,x)$ for all $y \in Y$. So we can assume without loss of generality that our assumptions hold for the product of two probability measures.

*Proof of Lemma 1.* Let $\Delta_\pi(\mathbb{R}\times A)$ be the space of Borel probability measures on $\mathbb{R}\times A$ with $\mathbb{R}$-marginal $\pi$. We define $\phi : \Delta(L) \to \Delta_\pi(\mathbb{R}\times A)$ by

$$\phi_\mu(B) = \int\int 1_B(p, l(p))\,\mathrm{d}\pi(p)\,\mathrm{d}\mu(l)$$

for each Borel set $B \subseteq \mathbb{R}\times A$. It follows from Balder (1981, Theorem 7.1) or the results in Ghoussoub (1982, Section I) that $\phi$ is a surjection. Moreover,

$$\begin{aligned} V(\mu,y) &= \int\int v\big(p,l(p),x\big)h(p,x,y)\,d\pi\otimes\xi(p,x)\,\mathrm{d}\mu(l) \\ &= \int\int\int v\big(p,l(p),x\big)h(p,x,y)\,\mathrm{d}\pi(p)\,\mathrm{d}\mu(l)\,\mathrm{d}\xi(x) \\ &= \int\int v\big(p,a,x\big)h(p,x,y)\,\mathrm{d}\phi_\mu(p,a)\,\mathrm{d}\xi(x), \end{aligned}$$

so we can study undominated mixed limit orders in terms of $\Delta_\pi(\mathbb{R}\times A)$. We can identify $\Delta_\pi(\mathbb{R}\times A)$ with a convex and compact subset of a locally convex Hausdorff topological vector space as in Balder (1988) by endowing $\Delta_\pi(\mathbb{R}\times A)$ with the narrow topology on Young measures. It follows from the Scorza-

Dragoni Theorem, see Denkowski, Migórski, and Papageorgiou (2003, Theorem 2.5.19), that this topology coincides with the usual topology of weak convergence of measures.

We can then, abusing notation a bit, treat $V$ as a continuous function $V : \Delta_\pi(\mathbb{R} \times A) \times Y \to \mathbb{R}$. We can also identify $Y$ homeomorphically with a closed subset of the weak*-compact set $\Delta(Y)$ via the embedding $y \mapsto \delta_y$. Abusing notation a bit more, we can extend $V$ to a bilinear function $V : \Delta_\pi(\mathbb{R} \times A) \times \Delta(Y)$. $V$ thus extended is continuous by Balder (1988, Theorem 2.5). By Phelps (2001, Proposition 1.2), $\Delta(Y)$ is, under the embedding, the closed convex hull of $Y$. So by Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2016, Lemma 1), an element $\tau$ of $\Delta_\pi(\mathbb{R} \times A)$ is undominated if and only if $\tau \in \operatorname{argmax} V(\cdot, \rho)$ for some $\rho \in \Delta(Y)$. □

*Proof of Proposition 1.* One direction is trivial. For the other direction, assume that $\mu'$ strictly dominates $\mu$. By the Berge maximum theorem, the function $\kappa \mapsto \min_y V(\kappa, y) - V(\mu, y)$ is continuous. By assumption, it achieves a strictly positive value at $\mu'$. To finish the proof, we make use of the fact that the set of deterministic limit orders, embedded via the function $l \mapsto \phi_{\delta_l}$, is dense in $\Delta_\pi(\mathbb{R} \times A)$ when $\pi$ is nonatomic. This denseness is familiar from the optimal control literature, the classic reference being Warga (1972, Theorem IV.2.6; 6). By this denseness, there exists a deterministic limit order $l$ such that $\min_y V(l, y) - V(\mu, y) > 0$. Then $\mu$ is dominated by the deterministic limit order $l$. □

*Proof of Theorem 2.* Let $\mu$ be undominated. By Proposition 1, there exists $\rho \in \Delta(Y)$ such that $\mu$ is a maximizer of $\int V(\cdot, y)\, \mathrm{d}\rho(y)$. For this to be possible, $\mu$ almost all $l$ must be maximizers of $\int V(\cdot, y)\, \mathrm{d}\rho(y)$. Such $l$ must prescribe a conditional best response at $\rho_P$-almost every price $p$. If $\int \int h(p, x, y)\, \mathrm{d}\xi(x)\, \mathrm{d}\rho(y) < p$, the unique best response is $s$ and if $\int \int h(p, x, y)\, \mathrm{d}\xi(x)\, \mathrm{d}\rho(y) > p$, the unique best response is $b$. □

*Proof of Theorem 3.* Let $\mu \in \Delta(L)$ be undominated. Since $\mu$ is undominated there must be, by Lemma 1, some $\rho \in \Delta(Y)$ such that $\mu$ maximizes

$\int V(\cdot, y)\, \mathrm{d}\rho(y)$. Now

$$\int V(\mu, y)\, \mathrm{d}\rho(y) = \int\int\int v\big(p, l(p), x\big) h(p,x,y)\; d\pi \otimes \xi(p,x)\; \mathrm{d}\mu(l)\; \mathrm{d}\rho(y)$$
$$= \int\int\int\int v\big(p, l(p), x\big) f(p) g(x,y)\; \mathrm{d}\xi(x)\; \mathrm{d}\rho(y)\; d\pi(p)\; \mathrm{d}\mu(l).$$

By a standard argument, for this to be maximal $l$ must maximize

$$\int\int\int v\big(p, l(p), x\big) f(p) g(x,y)\; \mathrm{d}\xi(x)\; \mathrm{d}\rho(y)\; d\pi(p)$$
$$= \int f(p) \int\int v\big(p, l(p), x\big) g(x,y)\; \mathrm{d}\xi(x)\; \mathrm{d}\rho(y)\; d\pi(p)$$

for $\mu$-almost all $l$ and for such $l$, for similar reasons, $l(p)$ must then maximize

$$\int\int v\big(p, l(p), x\big) f(p) g(x,y)\; \mathrm{d}\xi(x)\; \mathrm{d}\rho(y)$$

for $\pi$-almost all $p$. But this can be checked by going though the cases, the value of the double integral is

$$\int\int x \cdot g(x,y)\; \mathrm{d}\xi(x)\; \mathrm{d}\rho(y) - p$$

for $l(p) = b$,

$$p - \int\int x \cdot g(x,y)\; \mathrm{d}\xi(x)\; \mathrm{d}\rho(y)$$

for $l(p) = s$, and 0 for $l(p) = n$. But this forces $l$ to be the threshold limit order with threshold $\int\int x \cdot g(x,y)\; \mathrm{d}\xi(x)\; \mathrm{d}\rho(y)$. In particular, $\mu$ must choose the limit order $l$ with probability one and therefore be the threshold limit order with threshold $\int\int x \cdot g(x,y)\; \mathrm{d}\xi(x)\; \mathrm{d}\rho(y)$. □

# References


Francis J. Anscombe and Robert J. Aumann. A definition of subjective probability. *Annals of Mathematical Statistics*, 34:199–205, 1963.

Robert J. Aumann. Borel structures for function spaces. *Illinois Journal of Mathematics*, 5:614–630, 1961.

Erik J. Balder. *Mathematical foundations of statistical decision theory: the modern viewpoint I*. Univ. Utrecht, Department of Mathematics, 1981.

Erik J. Balder. Generalized equilibrium results for games with incomplete information. *Mathematics of Operations Research*, 13(2):265–276, 1988.

Pierpaolo Battigalli, Simone Cerreia-Vioglio, Fabio Maccheroni, and Massimo Marinacci. A note on comparative ambiguity aversion and justifiability. *Econometrica*, 84(5):1903–1916, 2016.

Pierpaolo Battigalli, Simone Cerreia-Vioglio, Fabio Maccheroni, and Massimo Marinacci. Mixed extensions of decision problems under uncertainty. *Economic Theory*, 63(4):827–866, 2017.

Patrick Beissner and Frank Riedel. Equilibria under Knightian price uncertainty. *Econometrica*, 87(1):37–64, 2019.

Truman F. Bewley. Knightian decision theory. Part I. *Decisions in Economics and Finance*, 25(2):79–110, 2002.

Simone Cerreia-Vioglio, Fabio Maccheroni, Massimo Marinacci, and Luigi Montrucchio. Uncertainty averse preferences. *Journal of Economic Theory*, 146:1275–1330, 2011.

Alain Chateauneuf and José Heleno Faro. Ambiguity through confidence functions. *Journal of Mathematical Economics*, 45(9-10):535–558, 2009.

Zdzisław Denkowski, Stanisław Migórski, and Nikolas S. Papageorgiou. *An introduction to nonlinear analysis: Theory*. Kluwer Academic Publishers, Boston, MA, 2003. ISBN 0-306-47392-5.

James Dow and Sérgio Ribeiro da Costa Werlang. Uncertainty aversion, risk aversion, and the optimal choice of portfolio. *Econometrica*, 60(1):197–204, 1992.

Richard M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. ISBN 0-521-00754-2. Revised reprint of the 1989 original.

David Easley and Maureen O'Hara. Ambiguity and nonparticipation: The role of regulation. *The Review of Financial Studies*, 22(5):1817–1843, 2009.

Jürgen Eichberger, Simon Grant, and David Kelsey. Randomization and dynamic consistency. *Economic Theory*, 62(3):547–566, 2016.

Daniel Ellsberg. Risk, ambiguity, and the Savage axioms. *The Quarterly Journal of Economics*, 75:643–669, 1961.

Larry G. Epstein and Martin Schneider. Ambiguity and asset markets. *Annual Review of Financial Economics*, 2(1):315–346, 2010.

Larry G. Epstein and Tan Wang. Intertemporal asset pricing under Knightian uncertainty. *Econometrica*, 62(2):283–322, 1994.

Peter C. Fishburn. *Utility theory for decision making*. Publications in Operations Research, No. 18. John Wiley & Sons, Inc., New York-London-Sydney, 1970.

Nassif Ghoussoub. An integral representation of randomized probabilities and its applications. In *Seminar on Probability, XVI*, volume 920 of *Lecture Notes in Math.*, pages 519–543. Springer, Berlin-New York, 1982.

Itzhak Gilboa and Massimo Marinacci. Ambiguity and the bayesian paradigm. In Daron Acemoglu, Manuel Arellano, and Eddie Dekel, editors, *Advances in Economics and Econometrics: Tenth World Congress*, volume 1 of *Econometric Society Monographs*, pages 179—-242. Cambridge University Press, 2013.

Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

Itzhak Gilboa, Andrew W. Postlewaite, and David Schmeidler. Probability and uncertainty in economic modeling. *Journal of Economic Perspectives*, 22(3):173–88, September 2008.

Lars Peter Hansen and Thomas J Sargent. Robust control and model uncertainty. *American Economic Review*, 91(2):60–66, 2001.

Shaowei Ke and Qi Zhang. Randomization and ambiguity aversion. Working paper, 2017.

Peter Klibanoff, Massimo Marinacci, and Sujoy Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849–1892, 2005.

Christoph Kuzmics. Abraham Wald's complete class theorem and Knightian uncertainty. *Games and Economic Behavior*, 104:666–673, 2017.

Fabio Maccheroni, Massimo Marinacci, and Aldo Rustichini. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447–1498, 2006.

Paul R. Milgrom and Robert J. Weber. Distributional strategies for games with incomplete information. *Mathematics of Operations Research*, 10(4): 619–632, 1985.

David G. Pearce. Rationalizable strategic behavior and the problem of perfection. *Econometrica*, 52:1029–50, 1984.

Robert R. Phelps. *Lectures on Choquet's theorem*, volume 1757 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 2001. ISBN 3-540-41834-2.

Howard Raiffa. Risk, ambiguity, and the Savage axioms: Comment. *The Quarterly Journal of Economics*, 75(4):690–694, 1961.

Luca Rigotti and Chris Shannon. Uncertainty and risk in financial markets. *Econometrica*, 73(1):203–243, 2005.

Luca Rigotti and Chris Shannon. Sharing risk and ambiguity. *Journal of Economic Theory*, 147(5):2028–2039, 2012.

Kota Saito. Preferences for flexibility and randomization under uncertainty. *American Economic Review*, 105(3):1246–1271, 2015.

David Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.

Kyoungwon Seo. Ambiguity and second-order belief. *Econometrica*, 77:1575–1605, 2009.

Maxwell B. Stinchcombe. Balance and discontinuities in infinite games with type-dependent strategies. *Journal of Economic Theory*, 146(2):656–671, 2011.

Abraham Wald. An essentially complete class of admissible decision functions. *The Annals of Mathematical Statistics*, 18(4):549–555, 1947.

Jack Warga. *Optimal control of differential and functional equations*. Academic Press, New York-London, 1972.

## Graz Economics Papers

For full list see:
http://ideas.repec.org/s/grz/wpaper.html
Address: Department of Economics, University of Graz,
Universitätsstraße 15/F4, A-8010 Graz

---

03–2019 **Michael Greinecker and Christoph Kuzmics**: Limit Orders under Knightian Uncertainty

02–2019 **Miriam Steurer and Robert J. Hill**: Metrics for Measuring the Performance of Machine Learning Prediction Models: An Application to the Housing Market

01–2019 **Katja Kalkschmied and Jörn Kleinert**: (Mis)Matches of Institutions: The EU and Varieties of Capitalism

21–2018 **Nathalie Mathieu-Bolh and Ronald Wendner**: We Are What We Eat: Obesity, Income, and Social Comparisons

20–2018 **Nina Knittel, Martin W. Jury, Birgit Bednar-Friedl, Gabriel Bachner and Andrea Steiner**: The implications of climate change on Germanys foreign trade: A global analysis of heat-related labour productivity losses

19–2018 **Yadira Mori-Clement, Stefan Nabernegg and Birgit Bednar-Friedl**: Can preferential trade agreements enhance renewable electricity generation in emerging economies? A model-based policy analysis for Brazil and the European Union

18–2018 **Stefan Borsky and Katja Kalkschmied**: Corruption in Space: A closer look at the world's subnations

17–2018 **Gabriel Bachner, Birgit Bednar-Friedl and Nina Knittel**: How public adaptation to climate change affects the government budget: A model-based analysis for Austria in 2050

16–2018 **Michael Günther, Christoph Kuzmics and Antoine Salomon**: A Note on Renegotiation in Repeated Games [Games Econ. Behav. 1 (1989) 327360]

15–2018 **Meng-Wei Chen, Yu Chen, Zhen-Hua Wu and Ningru Zhao**: Government Intervention, Innovation, and Entrepreneurship

14–2018 **Yu Chen, Shaobin Shi and Yugang Tang**: Valuing the Urban Hukou in China: Evidence from a Regression Discontinuity Design in Housing Price

13–2018 **Stefan Borsky and Christian Unterberger**: Bad Weather and Flight Delays: The Impact of Sudden and Slow Onset Weather Events

12–2018 **David Rietzke and Yu Chen**: Push or Pull? Performance-Pay, Incentives, and Information

11–2018 **Xi Chen, Yu Chen and Xuhu Wan**: Delegated Project Search

10–2018 **Stefan Nabernegg, Birgit Bednar-Friedl, Pablo Muñoz, Michaela Titz and Johanna Vogel**: National policies for global emission reductions: Effectiveness of carbon emission reductions in international supply chains

09–2018 **Jonas Dovern and Hans Manner**: Order Invariant Tests for Proper Calibration of Multivariate Density Forecasts

08–2018 **Ioannis Kyriakou, Parastoo Mousavi, Jens Perch Nielsen and Michael Scholz**: Choice of Benchmark When Forecasting Long-term Stock Returns

07–2018 **Joern Kleinert**: Globalization Effects on the Distribution of Income

06–2018 **Nian Yang, Jun Yang and Yu Chen**: Contracting in a Continuous-Time Model with Three-Sided Moral Hazard and Cost Synergies

05–2018 **Christoph Kuzmics and Daniel Rodenburger**: A case of evolutionary stable attainable equilibrium in the lab

04–2018 **Robert J. Hill, Alicia N. Rambaldi, and Michael Scholz**: Higher Frequency Hedonic Property Price Indices: A State Space Approach

03–2018 **Reza Hajargasht, Robert J. Hill, D. S. Prasada Rao, and Sriram Shankar**: Spatial Chaining in International Comparisons of Prices and Real Incomes

02–2018 **Christoph Zwick**: On the origin of current account deficits in the Euro area periphery: A DSGE perspective

01–2018 **Michael Greinecker and Christopher Kah**: Pairwise stable matching in large economies

15–2017 **Florian Brugger and Jörn Kleinert**: The strong increase of Austrian government debt in the Kreisky era: Austro-Keynesianism or just stubborn forecast errors?

14–2017 **Jakob Mayer, Gabriel Bachner and Karl W. Steininger**: Macroeconomic implications of switching to process-emission-free iron and steel production in Europe

13–2017 **Andreas Darmann, Julia Grundner and Christian Klamler**: Consensus in the 2015 Provincial Parliament Election in Styria, Austria: Voting Rules,Outcomes, and the Condorcet Paradox

12–2017 **Robert J. Hill, Miriam Steurer and Sofie R. Waltl**: Owner Occupied Housing in the CPI and Its Impact On Monetary Policy During Housing Booms and Busts

11–2017 **Philipp Kohlgruber and Christoph Kuzmics**: The distribution of article quality and inefficiencies in the market for scientific journals

10–2017 **Maximilian Goedl**: The Sovereign-Bank Interaction in the Eurozone Crisis

09–2017 **Florian Herold and Christoph Kuzmics**: The evolution of taking roles

08–2017 **Evangelos V. Dioikitopoulos, Stephen J. Turnovsky and Ronald Wendner**: Dynamic Status Effects, Savings, and Income Inequality

07–2017 **Bo Chen, Yu Chen and David Rietzke**: Simple Contracts under Observable and Hidden Actions

06–2017 **Stefan Borsky, Andrea Leiter and Michael Paffermayr**: Product Quality and Sustainability: The Effect of International Environmental Agreements on Bilateral Trade

05–2017 **Yadira Mori Clement and Birgit Bednar-Friedl**: Do Clean Development Mechanism projects generate local employment? Testing for sectoral effects across Brazilian municipalities

04–2017 **Stefan Borsky, Alexej Parchomenko**: Identifying Phosphorus Hot Spots: A spatial analysis of the phosphorus balance as a result of manure application

03–2017 **Yu Chen, Yu Wang, Bonwoo Koo**: Open Source and Competition Strategy Under Network Effects

02–2017 **Florian Brugger**: The Effect of Foreign and Domestic Demand on U.S. Treasury Yields

01–2017 **Yu Chen**: On the Equivalence of Bilateral and Collective Mechanism Design

13–2016 **Jan de Haan, Rens Hendriks, Michael Scholz**: A Comparison of Weighted Time-Product Dummy and Time Dummy Hedonic Indexes

12–2016 **Katja Kalkschmied**: Complementary Institutional Elements and Economic Outcomes

11–2016 **Dieter Balkenborg, Josef Hofbauer, Christoph Kuzmics**: The refined best reply correspondence and backward induction

10–2016 **Karl W. Steininger, Wolf D. Grossmann, Iris Grossmann**: The economic value of dispatchable solar electricity: a Post-Paris evaluation

09–2016 **Sofie R. Waltl**: Estimating aggregate quantile-specific gross rental yields for residential housing in Sydney

08–2016 **Karl Farmer**: Financial integration and house price dynamics in equilibrium modeling of intra-EMU and global trade imbalances

07–2016 **Gabriel Bachner, Birgit Bednar-Friedl**: Counterbalancing the Effects of Climate Change Adaptation on Public Budgets: Factor Taxes, Transfers, or Foreign Lending?

06–2016 **Margareta Kreimer, Ricardo Mora**: Segregated Integration: Recent Trends in the Austrian Gender Division of Labor

05–2016 **Karl Farmer, Bogdan Mihaiescu**: Credit constraints and differential growth in equilibrium modeling of EMU and global trade imbalances

04–2016 **Daniel Eckert, Frederik Herzberg**: The birth of social choice theory from the spirit of mathematical logic: Arrow's theorem in the framework of model theory

03–2016 **Christoph Kuzmics, Jan-Henrik Steg**: On public good provision mechanisms with dominant strategies and balanced budget

02–2016 **Florian Gauer, Christoph Kuzmics**: Cognitive Empathy in Conflict Situations

01–2016 **Thomas Aronsson, Olof Johansson-Stenman, Ronald Wendner**: Redistribution Through Charity, and Optimal Taxation when People are Concerned with Social Status